From M–Theory to F–Theory, with Branes

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Abstract

A duality relationship between certain brane configurations in type IIA and type IIB string theory is explored by exploiting the geometrical origins of each theory in M–theory. The configurations are dual ways of realising the non–perturbative dynamics of four dimensional $\mathcal{N}{=}2$ supersymmetric $SU(2)$ gauge theory with four or fewer flavours of fermions in the fundamental, and the spectral curve which organizes these dynamics plays a prominent role in each case. This is an illustration of how non–trivial F–theory backgrounds follow from M–theory ones, hopefully demystifying somewhat the origins of the former.

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1. Introduction.

1.1. Motivation

Certain configurations of extended objects in string theory have become of considerable interest of late, as they enable the intricate interplay of duality, geometry, field theory and string theory to be explored. Typically, these configurations involve combinations[1,2] of D–branes[3] and NS–(five)branes[4], and sometimes the inclusion[5,6] of orientifolds[7]. The field theories are realized in the dimensions common to all of the world–volumes of the extended objects in question. The dynamics of the field theories encode much of the geometrical behaviour of the branes and *vice–versa*, yielding a powerful laboratory for the study of familiar dualities and the discovery of new ones.

These configurations are still somewhat novel, and many of their properties remain to be fully understood. The aspects which we will study in this paper are concerned with the question of how the physics —as encoded in the world–volume field theory— of a given configuration can arise from a very different configuration of extended objects. We are thus studying a sort of 'dual pair' realizing the same field theory, together with the properties of the transformation between the members of the pair.

Consider for a moment the properties of 'T–duality', acting on closed string backgrounds. In the target geometry we can replace a circle of radius R by one of radius α'/R, where α' is the inverse string tension. When the background fields have no non–trivial dependence on the compact coordinate (at least asymptotically), we understand what happens very well: winding and momentum modes exchange roles, leaving the physics invariant. (Of course, examining the action on space–time fermions, we see that the type IIA string theory is exchanged with the type IIB.)

However in the open string sector, T–duality exchanges free boundary conditions on the string endpoints with fixed ones (while exchanging the circles), changing a Dp–brane into a D$(p+1)$–brane or *vice–versa*. Therefore, T–duality applied to the multi–brane configurations along[1] one of the dimensions containing the field theory will change the dimension of the field theory. This is *not* the type of transformation which we wish to consider. We wish to find a transformation on the configuration which leaves the physical content of the field theory invariant, including its dimensionality. As a result we must consider transformations along a direction in which some branes are extended and some branes are localized.

Necessarily therefore, we will study a transformation of the brane configuration which is essentially a complicated version of T–duality. 'Complicated' because it will involve two

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[1] Here, 'T–duality along a direction' will mean the following process in string theory: Compactify the direction on a circle and shrink the radius of the circle to zero. When T–duality applies, this process is equivalent to growing a new non–compact direction, giving the 'T–dual' configuration.

situations where T–duality —as phrased above— is not well understood:

(i) It will involve a direction along which the background fields (such as the dilaton, metric and Kalb–Ramond field, all from the Neveu–Schwarz/Neveu–Schwarz (NS–NS) sector) have non–trivial dependence, because an NS–brane has its core there.

(ii) It will involve a direction along which the world–volume of a D–brane is only of finite extent, because the D–brane ends on the NS–brane. (This latter situation can be interpreted as a non–trivial dependence of the Ramond–Ramond (R–R) background fields on the coordinate in question.)

The end result of establishing the transformation will be a realization of the *same field theory* by either a brane configuration in type IIA string theory or a brane configuration in type IIB string theory. As in each configuration the dilaton (and hence the respective string couplings) varies from place to place in space–time, it is more precise to say that we have a dual realization involving M–theory and F–theory backgrounds.

1.2. Rephrasing T–Duality using M–Theory.

The previous statement is the key to understanding just how we will proceed. In constructing the duality, we cannot use the strict definition of T–duality given above at all stages, as it is tied very much to the specific string theory context where the background field dependence is relatively trivial. Note however, that for very simple backgrounds we already know how we can embed our understanding of T–duality between type IIA and type IIB string theory into a larger context. First, recall that[8,9]:

(a) Ten dimensional type IIA string theory is the zero radius limit of M–theory[2] compactified on a circle.

Placing type IIA on a circle and shrinking it to zero size, we have by T–duality, an equivalent description in terms of ten dimensional type IIB string theory. The extra dimension is just the 'T–dual' dimension, which we understand very well in a stringy context as the infinite radius circle dual to the one of zero radius upon which the type IIA theory is compactified.

Thinking of this two–step process as a single operation on M–theory we arrive at the following conclusion[10,11]:

(b) Ten dimensional type IIB string theory is the zero size limit of M–theory compactified on a torus.

We will thus reinterpret T–duality between type IIA and type IIB string theory as those statements about how to arrive at each theory from M–theory.

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[2] It will suffice in the present context to take M–theory to mean 'eleven dimensional supergravity'. Strictly speaking, this is merely the low–energy limit of M–theory, whatever it turns out to be.

1.3. Geometrical Origins of Branes and F–Theory.

Nearly all of the D–branes in either theory have a simple understanding in terms of the above geometrical statements (*(a)* and *(b)*) together with the fact that M–theory contains two basic branes, the M2–brane[12] and the M5–brane[13].

In type IIA string theory, the D2–brane is a direct descendant of the M2–brane, while the D4–brane is the double reduction of the M5–brane[14], one dimension being wrapped on the circle. The F1–brane (*i.e.,* the fundamental type IIA string) is the double reduction of the M2–brane[15], while the F5–brane (NS–brane) is the direct descendant of the M5–brane. The D0–brane and D6–brane have a Kaluza–Klein origin as electric and magnetic sources[8,9].

Meanwhile, in the type IIB string theory, the D1–brane and the F1–brane come from wrapping one dimension of the M2–brane entirely on one or other cycle of the T^2 [16]. Similarly, the D5–brane and the F5–brane come from wrapping a dimension of the M5–brane on one or the other cycle of the T^2. These partial wrappings explain why the respective D– and F–branes are mapped into each other under the $\tau \rightarrow -1/\tau$ transformation of T^2 which exchanges the two cycles. Labelling them with integers (0,1) and (1,0) respectively, the full $SL(2,\mathbb{Z})$ non–perturbative symmetry produces a family of (p,q) branes[11,17]. The D3–brane comes from wrapping two dimensions of the M5–brane on the T^2, which explains[16] why it is mapped to itself under $SL(2,\mathbb{Z})$.

Understanding the existence of D7–branes in this geometrical picture is the launching point for understanding the origins of F–theory[18]. There, the configuration of seven–branes in the non–perturbative type IIB theory is given by the degeneration of an auxiliary torus fibred over the ten physical dimensions of the theory. The origin of this auxiliary torus is clear in the context of this discussion. Once we have arrived at the type IIB string theory (using *(b)* above), we must not forget the torus upon which we compactified M–theory. We shrunk the area of the torus but we had a choice about the complex tructure, τ. Indeed, the type IIB theory 'remembers' the complex structure of the torus, and this is frozen into the resulting configuration. Im(τ) is identified[11,10,18] with the inverse type IIB coupling $\lambda_B^{-1} = e^{-\Phi}$, ($\Phi$ is the dilaton field), while Re(τ) is the R–R scalar field $A^{(0)}$. The degeneration of the auxiliary torus fibration is a jump in the value of $A^{(0)}$, which signals the presence of a magnetic source of it, a seven–brane. There is a (p,q) family of these branes too, related by $SL(2,\mathbb{Z})$, and the $(0,1)$ member of this family is the D7–brane of perturbative type IIB string theory.

We will take the position here that this is the geometrical origin of F–theory: An elliptic fibration, defining a consistent type IIB background, is simply a concise way of specifying consistently a collection of data about a *family of tori* upon which M–theory has been compactified before ultimately shrinking them away.

In M–theory, the D6–brane is a Kaluza–Klein monople[8], which from a ten dimensional

point of view is a circle fibration which degenerates over the position of the D6–brane. This family of circles becomes part of the family of tori which specify the data in F–theory, as we will see. The degeneration of the circles (from the ten dimensional point of view) — signalling the presence of D6–branes in type IIA— are inherited by the tori, ultimately indicating the presence of D7–branes in type IIB. We will also see how other structures in type IIA/M–theory give rise to some of the other types of seven–brane of type IIB/F–theory. In this way, we see that F–theory backgrounds are simply a subset of the possible M–theory compactifications.

1.4. Beyond Simple T–Duality.

So far, we have employed rather heavy machinery to carry out a task which we can perform with simpler and sharper tools. We have recalled the rephrasing of T–duality and the taxonomy of branes in terms of the geometry of M–theory. We already understand T–duality very well in the terms laid out earlier, concerning the momentum and winding modes of closed strings, and boundary conditions for open strings.

However, the simple geometric restating of T–duality reiterated here is more readily adaptable to generalisation than the original terminology. Indeed, we should be able to incorporate features which we do not know how to handle well in the purely stringy context and we will do so in what follows.

We can proceed to understand relationships between non–trivial brane configurations in type IIA and brane configurations in type IIB as follows: Interpret the type IIA brane configuration as an M–theory background. This renders harmless many features which are hard to handle in string theory (such as branes ending on other branes) by turning them into smooth M–theory configurations[19,5]. Next, compactify that M–theory background upon a family of tori, chosen in a way which respects the symmetries of the brane configuration, and shrink the tori. The resulting background will be an F–theory background, corresponding to a type IIB configuration of extended objects with non–trivial NS–NS and R–R background fields given by the data of the shrunken tori.

Thus, the real use of the technique will become apparent when we try to study the analogues of T–duality in directions where there is non–trivial behaviour. The route described above will allow us to realize an effective duality transformation which would have been more difficult to determine using purely stringy techniques alone.

1.5. Outline

The plan of this paper is as follows. In section 2, we will start by describing the configuration of branes we wish to consider, in the type IIA string theory. It is essentially a review. Although it is a classical discussion, it is a good starting point to orient ourselves,

and it will sometimes be useful to return to the classical ten dimensional description for guidance.

In section 3, we review and follow the observation made in refs.[19,5] that to go beyond the classical physics, it will be useful to go to a smooth description of the branes as a configuration in M–theory, recovering within the brane geometry the spectral curve[20] which controls the (Coulomb branch) dynamics of the field theory[3].

The detailed procedures for constructing such smooth descriptions were presented in ref.[19], and we follow that presentation quite closely, specializing to the case in hand, recovering the smooth M–theory configuration as an M5–brane with topology $\mathbb{R}^4 \times T^2$ in a multi–Taub–NUT geometry.

In section 4, we depart from what has gone before, walking the path from M–theory to F–theory while carrying over the data of the M5–brane/multi–Taub–NUT configuration. We arrive thus at section 5, describing the F–theory configuration we expect to arrive at. Indeed, the spectral curve for the field theory under consideration has been previously recognized[23] as controlling the dynamics of a seven–brane configuration in type IIB/F–theory, and we make contact with that description. It has also been pointed out[24] that the $\mathcal{N}$=2, four dimensional field theory arises naturally on the world–volume of a D3–brane probe moving around in the seven–brane geometry. In our case, the D3–brane probe arises naturally as the remains of the M5–brane we found in the M–theory: Its toroidal part was wrapped on a space–time torus, which was subsequently shrunken away.

In section 6 we discuss the type IIB string theory (*i.e.*, classical) limit of the F–theory background, revisiting the work of refs.[23,24], recognizing and interpreting certain aspects of the 'dual' type IIA configuration in the new context.

We close with some remarks in section 7.

2. The Type IIA Brane Configuration.

(This and the next section constitute a review —tailored to our needs— and are included in order to set the scene, establish a few conventions, and attempt a self–contained discussion.)

In this section the statements which we shall make will be essentially classical ones, based on treating the fluctuations of flat branes. We will revisit this configuration in section 3, taking into account the branes' deformations away from flatness caused by the forces they

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[3] It should be noted here that another type of situation where the spectral curves of $\mathcal{N}$=2 field theories have been identified with the geometry of branes has been presented in the literature. (See ref.[21], for the original work and ref.[22], for a recent extension.) This context of that work is somewhat different from the contexts of refs.[19,5] and this paper, in which the identification is made after continuing to M–theory.

exert on each other. As a result, the field theory content we will deduce will be only true classically also.

Let us start with the following brane configuration in type IIA string theory:

type	#	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS	2	—	—	—	—	—	—	•	•	•	•
D4	2	—	—	—	—	•	•	[—]	•	•	•
D6	N_f	—	—	—	—	•	•	•	—	—	—

Table 1.

In Table 1 (and in a similar one in section 6), a dash '—' represents a direction *along* a brane's world–volume while a dot '•' is transverse to it. For the special case of the D4–branes' x^6 direction, where a world–volume is a finite interval, we use the symbol '[—]'. (A '•' and a '—' in the same column indicates that one object is living inside the world–volume of the other in that direction, and so they can't avoid one another. Two '•'s in the same column reveal that the objects are point–like in that direction, and need not coincide in that direction, except for the specific case where they share identical values of that coordinate.)

In the configuration the D4–branes are stretched, in the x^6 direction, between the two NS–branes which are a distance $x_1^6 - x_2^6 = L_6$ apart, where $x_{1,2}^6$ denote the positions of the first and second NS–brane in the x^6 direction. The remaining dimensions of their world–volumes, and that of all other branes, are fully extended, filling the directions in which they lie.

Consider the directions common to the world–volumes of all of the branes. There is a four dimensional field theory living on this common space–time (with coordinates (x^0, x^1, x^2, x^3)). This field theory has $\mathcal{N}=2$ supersymmetry, as the 32 supercharges are reduced by half due to the presence of the NS–branes, and by a half again due to the presence of the D4–branes. The presence of the D6–branes does not break any more supersymmetries[1].

The (classical) field content of the four dimensional theory is easily determined by the usual D–brane calculus: The excitations of open strings stretching between the D4–branes ('4–4 strings') supply some of the fields in the theory. Fluctuations parallel to the world–volume supply a family of fields transforming as vectors under the $SO(1,3)$ Lorentz symmetry. These vectors form $U(2)$ gauge bosons (when the D4–branes are coincident). Excitations transverse to the world–volume represent the movement of the D4–branes. The D4–branes must share the same position as the NS–branes in order to stay tethered to them, and

therefore there are no fluctuations in the (x^6, x^7, x^8, x^9) directions. The only transverse fluctuations are therefore in the (x^4, x^5) directions which gives a set of complex massless scalars in the field theory. Taking into account their transformation properties under the gauge symmetry, it is clear that they form the complex adjoint scalar ϕ, which lives in the $\mathcal{N}{=}2$ vector multiplet. The strength of the gauge coupling g is a function of the distance between the NS–branes: $g^2 \propto \lambda_A/L_6$. Here, λ_A is the type IIA string coupling, appearing in this way because the gauge kinetic term arises in open string theory (*i.e.*, the D–brane sector) as a disc amplitude.

The 'matter' multiplets of the gauge theory are $N_f(\leq 4)$ families of 'quark': scalars in the fundamental of $U(2)$, which come from the '6–4 strings' connecting the D6–branes to the D4–branes. The masses of these quarks are set by the distance (in (x^4, x^5)) between the D6–branes and the D4–branes.

The Higgs branch of the theory is reached by first making the quarks massless by moving the D6–branes to be coincident with the D4–branes. The D4–branes may now split, letting them have new endpoints on the D6–branes, and the segments are now free to move independently inside the D6–branes' world–volumes. The (x^7, x^8, x^9) positions parameterize the vacuum expectation values ('vevs') of the quarks. In this way the gauge symmetry can be completely Higgsed away.

The Coulomb branch of the theory (our concern for most of the paper) is reached by giving the adjoint scalar ϕ a vev, with values in the Abelian subalgebra of $U(2)$. This breaks the gauge symmetry down to $U(1){\times}U(1)$ and corresponds to moving the D4–branes apart in the (x^4, x^5) directions. When a D4–brane encounters a D6–brane in (x^4, x^5), a quark becomes massless.

We need to understand this complicated brane configuration much better. For example, the ending of the D4–branes on the NS–branes is a somewhat singular situation. One might expect this feature to be smoothed out in a way which corresponds to quantum corrections to the field theory statements we have made in this section. Ultimately, the geometry reproduces the structure of the spectral curves[20] which govern the structure of the quantum moduli space of the gauge theories under discussion. This was anticipated and exploited in ref.[5], and independently in ref.[19]. In ref.[19], the mechanisms by which the corrections to the brane configurations may be deduced were explained, and the consequences explored quite extensively.

3. The M–Theory Configuration.

The starting point for correcting our classical configuration of the previous section is to realize[19] that the definite position assigned the NS–branes in the x^6 direction is modified considerably. The D4–branes, which are finite in that direction and suspended between the NS–branes, are pulling the (x^4, x^5) portion of the NS–branes' world–volume out of

shape, giving asymptotically the shape of (say) the first NS–brane world–volume as:

$$x_1^6 = k \left(\ln |v - a_1| + \ln |v - a_2| \right) + \text{const.}, \qquad (3.1)$$

where $v = x^4 + ix^5$, and k is a constant which depends upon the string coupling. Here, a_1 and a_2 are the positions of the two D4–branes in the (x^4, x^5) plane.

In order for the NS–brane's kinetic energy integral

$$\int d^4x \, d^2v \sum_{\mu=0}^{3} \partial_\mu x^6 \partial^\mu x^6 \qquad (3.2)$$

to converge, we have

$$a_1 + a_2 = \text{C}, \qquad (3.3)$$

where C is some constant characteristic of the NS–brane. It can be set to zero after a shift of the origin in (x^4, x^5) space. As discussed before, the a positions are the scalar components of the gauge supermultiplet in the field theory. The sum $a_1 + a_2$ controls the overall $U(1)$ factor of the gauge group $U(2)$ and therefore equation (3.3) freezes out this $U(1)$, making our gauge group $SU(2)$. Considering the opposite D4–brane ends, on the other NS–brane, leads to the same equation and no additional conditions on the gauge group.

Turning to the gauge coupling, we revise our earlier formula to make it a function of v:

$$\frac{1}{g^2(v)} = \frac{x_1^6(v) - x_2^6(v)}{\lambda_A}, \qquad (3.4)$$

and so we see that it is behaving as it should for a gauge theory, varying as a function of some 'mass scale' set by $|v|$: the quantity $1/g^2$ diverges logarithmically as $|v| \to \infty$.

The next step is to recognize[5,19] that this type IIA situation of D4–branes ending on and deforming NS–branes should have a better description in M–theory. This is because on going to M–theory an extra dimension unfolds, revealing that there the D4–branes have a hidden world–volume dimension, and so become M5–branes. The NS–branes also become M5–branes, with a definite position in this new 'M–direction', x^{10}. The parts of the D4–branes we described in section 2 as lines in x^6 are actually cylinders connecting the NS–branes. The final justification for going to M–theory was pointed out in ref.[19]: Looking at formula (3.4), it is clear that if we increase the string coupling λ_A while simultaneously increasing the inter–NS–brane distance, the field theory is completely unaffected by this. Therefore, we can go to the M–theory limit, where we grow an extra dimension, x^{10}, of radius $R \sim \lambda_A^{2/3}$, as measured in type IIA units.

We now recognize[19] that the formulae above were the real part of a complex story. Giving the NS–branes positions in the x^{10} direction, we have:

$$x_1^6 + ix_1^{10} = R \left(\ln(v - a) + \ln(v + a) \right) + \text{const.}, \qquad (3.5)$$

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and we may define the coupling (measuring now in M–theory units of length[4])

$$\tau(v) = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2(v)} = \frac{(x_1^{10} - x_2^{10})}{2\pi R} + i\frac{(x_1^6(v) - x_2^6(v))}{2\pi R}. \qquad (3.6)$$

The angle θ changes harmlessly by $\pm 2\pi$ as an x^{10} position of an NS–brane changes by $2\pi R$, as it should.

We can quickly compute the β–function of our field theory using the above formula as follows: Following the arguments of ref.[1], made in the context of string theory (*i.e.*, the language of section 2), we know that we can move all of the D6–branes past one of the NS–branes (let us choose the second one), resulting in a D4–brane stretched from the NS–brane (starting on the other x^6–side of it from the gauge D4–branes) to a D6–brane, one for each D6–brane.

As the D6–branes are more massive than the D4–branes, 4–4 strings entirely in the new D4–brane sector do not contribute to the gauge group. However, the quarks are still present, as they now arise as N_f types of 4–4 string which connect the new D4–branes across the NS–brane to the old D4–branes. Since the D4–branes on the other side of the NS–brane pull the other way, the asymptotic shape of the NS–brane with the extra branes is given by:

$$x_2^6 + ix_2^{10} = R\left(\sum_{i=1}^{N_f} \ln(v - m_i) - \ln(v - a) - \ln(v + a)\right) + \text{const.}, \qquad (3.7)$$

where the m_i are the D6–brane (x^4, x^5) positions, or equivalently those of the new D4–branes. They are the classical masses of the quarks.

Looking at the large $|v|$ behaviour of the coupling using this formula, we get

$$2\pi i\tau(v) = -(4 - N_f)\ln v, \qquad (3.8)$$

displaying the one–loop β–function. When N_f=4 it vanishes, as it ought to for the scale invariant theory.

The way[19] to incorporate the D6–branes in this set–up directly in the M–theory picture is to recognize[8] that they are Kaluza–Klein monopoles[25]: The M–coordinate x^{10} is not simply a circle with which we form a product with the (x^4, x^5, x^6) directions to get the full space–time. Instead, it is fibred over them in a Hopf–like fashion. The metric geometry of this situation is that of multi–Taub–NUT. The positions of the D6–branes are the positions in the base where the Killing vector for translations in the x^{10} circle vanishes, giving us

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[4] Lengths measured in type IIA units, L_A, compare to lengths measured in M–theory units, L_M, by the formula $L_A = R^{1/2}L_M$.

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a singularity in the D6–brane metric when we reduce to ten dimensional type IIA string theory.

It is now clear that the type IIA string theory configurations of branes is a much less singular affair when viewed at strong coupling, in M–theory. The D4–branes and NS–branes are just different glimpses of the history of a single M5–brane's life–time. If we add a point representing infinity to the (x^4, x^5) world–volumes of the NS–branes, we see that in the full M–theory interpretation, the world–volume of the M5–brane has topology $\mathbb{R}^4 \times T^2$, where the T^2 is described as a surface embedded in the four dimensional space Q_{N_f}. Here, Q_{N_f} denotes the multi–Taub–NUT space of multiplicity N_f, the M–theory origin of the N_f D6–branes. In particular, Q_0 is just the product $\mathbb{R}^3 \times S^1$ with coordinates (x^4, x^5, x^6, x^{10}). As pointed out in ref.[19], it will suffice (for study of the Coulomb branch of the field theories) to represent Q_{N_f} as an equation of the form:

$$yz = \prod_{i=1}^{N_f}(v - m_i), \tag{3.9}$$

where (y, z, v) are coordinates on a three complex dimensional space with the structure of $\mathbb{C}^3$. As before, $v = x^4 + ix^5$. Defining the coordinate $s = (x^6 + ix^{10})/R$, we have that for fixed z, large y corresponds to $t = \exp(-s)$ while for fixed y, large z corresponds to t^{-1}. The parameters m_i are the (x^4, x^5) positions of the D6–branes. We will require that the N_f D6–branes are located *between* the NS–branes, and nowhere else. The specification (3.9) misses (among other things) the x^6 positions of the D6–branes.

The world–volume of the M5–brane may be specified as a further constraint equation in the coordinates (y, v): $F(y, v) = 0$. Giving Q_{N_f} a complex structure and requiring holomorphicity in v and y (very natural when viewed from the point of view of the field theory) specifies the metric structure on T^2 as a complex Reimann surface.

As a polynomial, the function F must be quadratic in y for a (v=const.) slice to yield two NS–branes in the ten dimensional picture, and our constraint equation is thus of the form[19]:

$$A(v)y^2 + B(v)y + C(v) = 0, \tag{3.10}$$

where A, B and C are relatively prime polynomials.

There are no components of D4–branes extended outside the $x_1^6 — x_2^6$ interval; these would necessarily be semi–infinite (as they have nothing else to end on), and as such would show up in our solution as a divergence in y for some definite value of v. The absence of such behaviour fixes A to be a constant, which we can choose to be 1. The same requirement also removes the possibility of z diverging for some particular value of v and this translates into a condition on the form of C and B also: C must have the same zeros –with the same multiplicity[5]– in the v plane as has the defining polynomial (3.9) of Q_{N_f}, and B must be quadratic in v in order to yield two D4–branes at fixed y in the ten dimensional picture.

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[5] This corresponds to placing all of the D6–branes *between* the NS–branes (in x^6), where they can

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Our torus is thus of the form:

$$y^2 + B(v)y + f \prod_{i=1}^{N_f}(v - m_i) = 0, \tag{3.11}$$

where f is an arbitrary complex constant. We can remove terms linear in v from $B(v)$ by a shift in v, which would shift the bare masses m_i. For the case $N_f=0$, the last term should simply be a constant, which we can set to 1 without loss of generality. In terms of $\tilde{y}=y+B/2$, we have:

$$\tilde{y}^2 = \frac{B(v)^2}{4} - f \prod_{i=1}^{N_f}(v - m_i) = 0, \tag{3.12}$$

a standard form for the spectral curve controlling the Coulomb branch of $\mathcal{N}=2$ supersymmetric four dimensional $SU(2)$ gauge theory with N_f quarks. The details of the polynomial can be fixed by comparing to various field theory limits as done in ref.[20].

4. A New Direction.

At the present stage, we have an M–theory background consisting of an M5–brane with topology $\mathbb{R}^4 \times T^2$ propagating in the N_f Taub–NUT space Q_{N_f}. The torus T^2 and the space Q_{N_f}, are all described in terms of constraint equations in an auxiliary six dimensional space.

Consider now the following. Let us ask instead for a slightly different situation, which will differ from this one in ways which are invisible in the field theory. Interpret the equation (3.10) as not only specifying the T^2 giving the shape of the M5–brane in the four dimensional space Q_{N_f}, but also specifying two of the space–time coordinates of the M–theory configuration. In other words, *we have wrapped the M5–brane we have been discussing on a space–time torus of the same shape.*

The manipulations following equation (3.9) and resulting in the final curve (3.12) serve to find us a smooth description of the wrapped M5–brane on a space–time torus T^2, where the torus is fibred over a base with topology $\mathbb{R}^2$. Some of the fibration data is inherited from that of the multi–Taub–NUT geometry: The information about the positions where the D6–branes live translates into a contribution to the information about the location of zeros of the discriminant of the torus fibration.

Let us return to the type IIA description for a moment. As the Kaluza–Klein monoples feel no forces amongst themselves, it is not problematic to have toroidally compactified one of the directions in which they are point–like. The wrapping of the M5–brane on

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do some good, instead of outside the interval, where they are irrelevant[19].

the torus is already partially performed from the start: the D4–branes are a piece of an M5–brane wrapped on the periodic x^{10} direction. So at any x^6 position where there is a D4–brane, we know that there is a hidden part of an M5–brane wrapped on x^{10}. What we have effectively done is a further compactification of eleven dimensional space–time. Focusing on the world–volume of an NS–brane, we must make some combination of (x^4, x^5) compact in order to get the complete toroidal topology. We know from our experience with the branes just how to do this: We simply add the space–time point at infinity to the (x^4, x^5) plane making it a $\mathbb{P}^1$, just as we did to the world–volume of the NS–branes in those directions. The $\mathbb{P}^1$ has cuts or punctures in it due to the presence of the D4–branes.

We have already seen[19] that the size of the M–direction does not affect the physics of the field theory if we rescale the separation of the NS–branes accordingly. Similarly, the fact that we have a $\mathbb{P}^1$ for the (x^4, x^5) direction (instead of $\mathbb{R}^2$) should not enter as a parameter in the field theory if we rescale the positions of the D4– and D6–branes to absorb any changes we make in the overall size of the $\mathbb{P}^1$.

Returning to M–theory where the complete, smooth description is to be found, we may now consider shrinking the T^2 part of the M5–brane wrapped space–time. We hold the complex structure of the torus (and hence the field theory data) fixed and shrink its size away to zero.

5. The F–Theory Configuration.

As described in the introduction, we know from simpler situations that we have a type IIB description of this situation (M–theory on a shrunken torus) where:

(i) We have a new direction, $\hat{x}$, which restores us to a ten dimensional theory.

(ii) The wrapped M5–brane becomes a D3–brane.

(iii) The data describing the shape of the torus which we shrink to zero size is not lost, but is 'remembered' by the final configuration: It is frozen into an auxiliary torus, fibred over the ten dimensions of the IIB theory. This is longhand for 'F–theory'.

As we know, the 'data torus', or more precisely the family of such tori, is that which specifies the Coulomb branch of the $\mathcal{N}{=}2$ four dimensional $SU(2)$ gauge theory with N_f quarks. Described as an elliptic fibration over a base $\mathcal{B}$, with topology $\mathbb{R}^2$, it is singular over up to six points (depending upon N_f) in $\mathcal{B}$. From the point of view of our F–theory background, these points are the positions of magnetic sources of the R–R background field $A^{(0)}$, as the modular parameter of the torus fibre specifies type IIB string background fields *via* the relation:

$$\tau(v) = A^{(0)}(v) + i\mathrm{e}^{-\Phi(v)}, \tag{5.1}$$

where the type IIB string coupling $\lambda_B(v)$ is related to the dilaton Φ as $\lambda_B=\mathrm{e}^\Phi$. Such a magnetic source is an object which is point–like in $\mathcal{B}$ and extended in the other eight directions. It is therefore a seven–brane of type IIB theory. In the case where we can describe the background with perturbative type IIB strings, the seven–brane is either a D7–brane or an O7–plane (orientifold fixed plane). More generally, it can be any of the infinite family of seven–branes which can appear in the type IIB theory by virtue of the $SL(2,\mathbb{Z})$ non–perturbative symmetry.

The connection between precisely this family of tori (3.12) (describing $D{=}4$, $\mathcal{N}{=}2$ $SU(2)$ gauge theory with N_f quarks) and an F–theory background was noticed in ref.[23]. It was pointed out there that close to the perturbative type IIB limit of F–theory compactified on K3 (*i.e.,* the orbifold limit of the K3), the background describes four identical families of six seven–branes. Focusing on one family, in the limit two of the six possible singularities merge to become an O7–plane while the rest become N_f D7–branes. Furthermore, the four dimensional field theory is naturally realized on the world–volume of a D3–brane probe, as pointed out in ref.[24]. The fact that the D3–brane has an $SU(2)$ living on it instead of just $U(1)$ is T–dual to the fact[26,27] that it is really *two* D3–branes, plus an orientifold projection which forces them to move together as a single object, projecting the expected $U(2)$ (resulting from their coincidence) to $SU(2)$.

We see here that the D3–brane probe appears unbidden in this framework as the wrapped M5–brane! We also know that the N_f D7–branes have their origins in the presence of N_f D6–branes, while the O7–plane is an additional structure which was frozen into the torus because of the non–trivial way (from the type IIA picture) the D4–branes end on the NS–branes. We can trace the origins of the O7–planes to the D4/NS–brane system and not the D6–branes because the case of no flavours has precisely two O7–planes and no other singularities (not counting the point at infinity).

In the next section we shall describe this further in the type IIB limit.

6. The Type IIB Brane Configuration.

Let us choose to label the coordinates of the base $\mathcal{B}$ by $v=x^4+ix^5$. (We should be careful here. This is not exactly the (x^4, x^5) pair of the type IIA configuration.) Let us also denote by $\hat{x}^6$ the new, 'dual direction' (which we briefly called $\hat{x}$ in the last section).

We have the following brane configuration in type IIB string theory:

type	#	x^0	x^1	x^2	x^3	x^4	x^5	$\hat{x}^6$	x^7	x^8	x^9
O7	2	—	—	—	—	•	•	—	—	—	—
D3	2	—	—	—	—	•	•	•	•	•	•
D7	N_f	—	—	—	—	•	•	—	—	—	—

Table 2.

Comparing Table 1 and Table 2, we see that from a string theory point of view we have performed a sort of T–duality, in the x^6 direction. As one might expect, under it the D6–branes have turned into D7–branes, as they should. Ignoring for a moment the finite extent of the D4–branes in the x^6 direction, we see that they have turned into a pair of D3–branes, as one might hope naively. The complication of the presence of the cores of two NS–branes, together with the ending of a D4–brane on them, turns out to be 'T_6–dual' to an orientifold background. The orientifold procedure glues to the two D3–branes into one dynamical object carrying an $SU(2)$ gauge group, and introduces an O7–plane.

This perturbative type IIB string background describes aspects of the classical limit of the Coulomb branch of the $SU(2)$ gauge theory. The position of the D3–brane in the (x^4, x^5) plane parameterizes the Coulomb branch of the gauge theory on its world volume, where the gauge group is generically $U(1)$. As it moves around the plane, it sees N_f D7–branes each of charge 1 (in D7–brane units), and one fixed plane, which is the O7–plane, the fixed plane of the orientifold symmetry, which is ΩR_{45} on the bosonic sector. If the D3–brane probe is coincident with the O7–plane, the $SU(2)$ is restored. (Here, Ω is world–sheet parity, and R_{45} is $v \rightarrow -v$. The O7–plane has charge -4 as can be deduced from requirements of $A^{(0)}$ charge cancellation in the full compact situation: In total there are four O7–planes and sixteen D7–branes.)

(As explained a while ago in ref.[24], this is understood in the T_{45}–dual type I language as follows: The D5–brane has gauge group $SU(2)$, resulting from a projection with Ω, in constructing the type I theory[26,27]. It has part of its world–volume in the directions (x^4, x^5) before doing the T_{45}–duality to the present situation. This allows the possibility of introducing (x^4, x^5) Wilson lines (when making them toroidal in preparation for the T–duality) to break the $SU(2)$ to $U(1)$. These Wilson lines are T_{45}–dual to the positions of the D3–brane probe here.)

Using the charge assignments just given, and the fact that the number of transverse directions is two, one expects[23] that the couplings are given by:

$$\tau(v) = \frac{1}{2\pi i} \left(\sum_{i=1}^{N_f} \ln(v - m_i) - 4 \ln v \right) + \text{const.}, \qquad (6.1)$$

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where m_i are the classical positions of the D7–branes and we have placed the O7–plane at the origin.

The similarity with the equations describing the asymptotic shape of the NS–branes as they are pulled on by the D4–branes (in section 3) should not escape our notice. Combining equations (3.5) and (3.7), we have (placing the D4–branes at the origin):

$$\tau(v) = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} = \frac{(x_2^{10} - x_1^{10})}{2\pi R} + i\frac{(x_1^6(v) - x_2^6(v))}{2\pi R} =$$
$$= \frac{1}{2\pi i}\left(\sum_{i=1}^{N_f} \ln(v - m_i) - 4\ln v\right) + \text{const.} \tag{6.2}$$

The m_i are the (x^4, x^5) positions of the D6–branes. The similarity between the two formulae is not an accident. It is part of the 'dual' properties of the brane configurations. Let us list some observations about these:

(i) In both cases there is $\mathcal{N}{=}2$ supersymmetry in four dimensions. The original thirty–two supercharges are reduced to eight. In the type IIA case this is done by introducing NS–branes, and then D4–branes. Adding D6–branes to the mix places no further constraints on the number of supercharges. Similarly, in the type IIB situation, there is a $\mathbb{Z}_2$ orientifold (which introduces an O7–plane), followed by the introduction of a D3–brane. Adding D7–branes to these does not 'break' any more supersymmetry.

(ii) In both cases, the logarithmic form of the two equations above is a consequence of there being two relevant directions in which a Laplace–Poisson equation is solved. In the type IIA situation, it is the two directions on the NS–brane in which the incident D4–branes make a point, pulling in a transverse x^6 direction. In the type IIB scenario, it is the two directions transverse to both the seven–branes and the D3–brane probe.

(iii) The main sources of non–trivial behaviour of the dilaton in the type IIA theory are the cores of the NS–branes, at the place where the D4–branes meets them. Equation (6.2) encodes the asymptotic shape of the NS–branes' world–volumes, deformed in the x^6 direction, and implicitly the distribution of background NS–NS and R–R fields there. The 'dual' configuration in type IIB makes this explicit: The D7–branes and O7–planes are NS–NS sources for the dilaton and R–R sources for the field $A^{(0)}$, and equation (6.1) gives their asymptotic form, while the branes themselves remain undeformed.

(iv) We can deduce that the D4/NS–brane system, non–trivial in the (x^4, x^5, x^6) sector, acts as an electric source for the R–R form $A^{(7)}$ in type IIA, and hence has some effective D6–brane charge, as measured by enclosing that part of the configuration with a two–sphere at infinity.

There are a number of ways to see that this is true:

(a) These charges are ultimately responsible for the O7–plane (two extra seven–branes) in the 'dual' type IIB (F–theory) picture. Interpreting our configurations as effectively

T_6–dual to each other, the O7–plane, carrying $A^{(0)}$ charge, is the image under T_6 of the D4/NS–brane junctions.

(b) This charge assignment is consistent with the fact that adding D6–branes, positioned precisely in the (x^4, x^5, x^6) directions, does not break any of the supersymmetries already preserved by the D4/NS–brane configuration. From the point of view of the D6–branes, adding them to the configuration is no different from adding them to a system of parallel D6–branes.

(c) Possessing electric charge of $A^{(7)}$ is equivalent to having some magnetic $A^{(1)}$ (D0–brane) charge. It is clear that the D4/NS–brane configuration has such charge by considering the nature of the x^6 end–point of the D4–brane in the (x^4, x^5) part of the NS–brane's world–volume: It is a 'vortex' or monopole. As one circles a D4–brane's end–point once in (x^4, x^5, x^6) space[6] and returns to the same position, some winding has been acquired in the x^{10} direction. This is the only way to make local sense of the smoothing out of the D4/NS–brane IIA system into a Reimann surface in M–theory[7]. This non–trivial winding is akin to the behaviour which we attribute to a D6–brane in assigning it the role of a Kaluza–Klein monopole of $A^{(1)}$.

(d) The $A^{(7)}$ charge observation is also consistent with the observation[1] that moving a D6–brane through an NS–brane will result in a new D4–brane stretched between them. Indeed, if we had moved the D6–branes off to infinity, obtaining the quarks from the resulting N_f semi–infinite D4–branes instead, the final equation for the shape of the M5–brane would have been precisely the same as the one obtained here, (3.12) [19]. Hence, the F–theory result would have been the same, and consequently so would be the final dual type IIB configuration in Table 2. Therefore, the effective T_6 duality treats the D4/NS–brane junction as an object with D6–brane charge.

(v) As pointed out in ref.[23], the equation (6.1) can only be correct classically, or far away from the O7–plane. Close to the orientifold, the imaginary part of τ would appear to be able to go negative, which is not acceptable in a theory which is supposed to be unitary. This is simply a reflection of the fact that there are non–perturbative corrections to the formula (6.1) as one approaches the orientifold. The full solution is obtained by returning to the complete F–theory background. The new non–perturbative data are precisely those encoded in the spectral curve (3.12), which yields the correct solution for τ everywhere and hence the non–perturbative positions of the seven–branes. An important fact is that the singularity at v_0, representing the O7–plane, splits into two pieces, separated by a distance $e^{\pi i \tau}$. This corresponds to the O7–plane splitting into two (p, q) seven–branes in the full non–perturbative theory. Similarly, the form (6.2) for the shape of the NS–branes is only true asymptotically; the complete data are in the M5–brane M–theory configuration

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[6] It is appropriate to think of the end–point as living in this space and not the (x^4, x^5) space alone once we take into account that the NS–brane is not at a definite x^6 position.

[7] I am grateful to M. Crescimanno and R. C. Myers for a conversation apiece, concerning this issue.

in the shape of the spectral curve (3.12).

Note that we can move from the theory with N_f=4 quarks to a lower number of quarks by the scaling limits described in ref.[20]. For example, we send a D7– (or D6–) brane (corresponding to a a quark of mass m) off to infinity in the (x^4, x^5) plane. At the same time, we take the limit $\tau \rightarrow i\infty$, and hold the product $\Lambda = e^{\pi i \tau} m$ fixed, defining the mass scale of the N_f<4 theory.

7. Closing Remarks.

We have found that a type IIA configuration of D4–branes, NS–branes and D6–branes on whose intersection there lives an $\mathcal{N}$=2 four dimensional $SU(2)$ gauge theory is related to a type IIB configuration of parallel D3–branes, D7–branes and O7–planes, realizing the same gauge theory. The spectral curve controlling the dynamics of the gauge theory appears naturally in the topology and geometry of M–branes in M–theory on the one side, and as F–theory data on the other.

We have studied a very non–trivial example of how F–theory brane configurations may arise as M–theory ones, realizing an effective T–duality in the process.

It seems that generalising the reverse process is always possible: We should be able to start with an F–theory background and shrink a direction over which the data torus is not varying much. This should yield an M–theory background where the torus has now become physical. If there were D3–branes in the F–theory background, they will become M5–branes with two of their dimensions in the shape of that torus. Returning to a type IIA background by shrinking an appropriate circle will yield a configuration of intersecting branes of various sorts. This procedure should always be possibly locally, and therefore we can understand (at least piece–wise) all F–theory backgrounds in terms of M–theory brane configurations.

The generalisation of the M–theory to F–theory route (along the lines of this paper) might be more challenging, however. It would be interesting to study how the example presented here might generalise, providing a useful relation between certain type IIA/M–theory brane configurations and (pieces of) type IIB/F–theory ones. There are many reasons why this would be desirable. Much of the technology of F–theory is very well organised in terms of the well–developed geometry of elliptically fibred complex manifolds. However, the study of complicated M–theory/type IIA brane configurations is still a relatively new area, so being able to relate them to F–theory backgrounds should help in sharpening certain aspects of their analysis.

However, it is not clear that all relevant M–theory brane configurations can be converted to F–theory ones in the specific way done here. Considering the case of higher rank gauge groups, where the spectral curves are of higher genus than that of a torus, is already

interesting: the path to F–theory will probably involve multiple wrappings of the M5–brane on the space–time torus, resulting in many D3–branes in the final dual model, with additional discrete projections.

It will be interesting to study such issues further. The benefits of finding a dictionary between M– and F–theory configurations will be of tremendous value in the study of the dynamics of gauge theories.
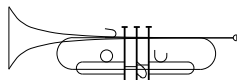
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